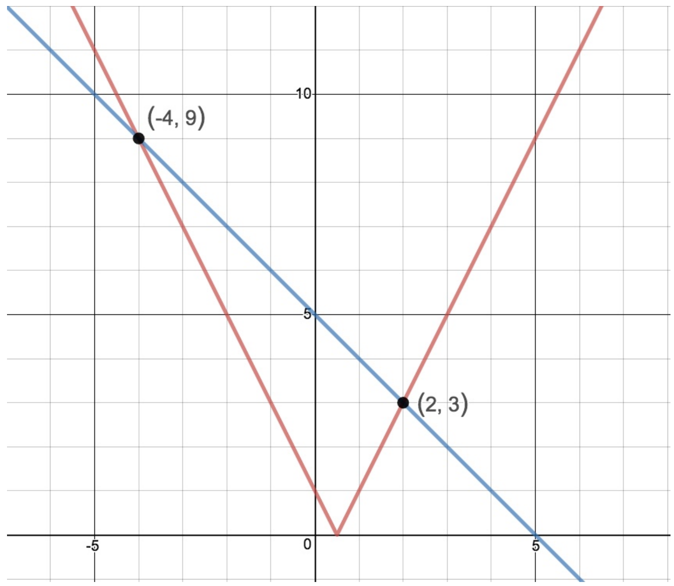
MATH 1650: SECTIONS 1.3 / 1.4: GRAPHICAL METHODS TO SOLVE EQUATIONS AND INEQUALITIES

In this section, we discuss methods of using graphs to help us solve equations and inequalities.

EXAMPLE: Below are graphed $f(x) = |2x - 1|$ and $g(x) = 5 - x$.



- Which graph is which? How do you know?

- Find $f(-4)$ and $g(-4)$ algebraically. What do you notice? How is this relationship reflected on the graph?

- Find $f(2)$ and $g(2)$ algebraically. What do you notice? How is this relationship reflected on the graph?

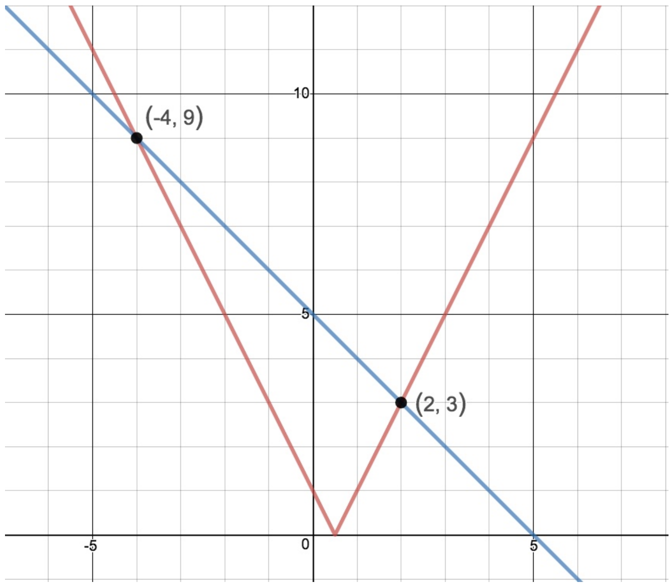
- Find $f(0)$ and $g(0)$. Which one is larger? How is this relationship shown on the graph?

- Find $f(4)$ and $g(4)$. Which one is larger? How is this relationship shown on the graph?

IN GENERAL: Suppose f and g are functions. The solutions to:

- $f(x) = g(x)$ are the x-values where the graphs of $y = f(x)$ and $y = g(x)$ **intersect**.
- $f(x) < g(x)$ are the x-values where the graph of $y = f(x)$ is **below** the graph of $y = g(x)$.
- $f(x) > g(x)$ are the x-values where the graph of $y = f(x)$ is **above** the graph of $y = g(x)$.

EXAMPLE: (Continued): Below are graphed $f(x) = |2x - 1|$ and $g(x) = 5 - x$.



- How does the graph above tell you there are only **two** solutions to $|2x - 1| = 5 - x$? What are they?

- Use the graph above to solve $|2x - 1| < 5 - x$. Write your answer in interval notation.

- Use the graph above to solve $|2x - 1| > 5 - x$. Write your answer in interval notation.

- Use the graph above to solve $|2x - 1| \geq 5 - x$. Write your answer in interval notation.

EXAMPLE: Suppose we wish to solve $|5x-3| \leq 6-2x$.

- Let's first solve the corresponding equation $|5x-3| = 6-2x$.

  HINT: To get started, note that if $|5x-3| = 6-2x$, then $5x-3 = 6-2x$ or $5x-3 = -(6-2x)$ ...

- Next, let's graph $f(x) = |5x-3|$ and $g(x) = 6-2x$ using a graphing utility to check your answers to $|5x-3| = 6-2x$.

- Use the first two parts of this problem to solve write down the solution to: $|5x-3| \leq 6-2x$. Use interval notation.

IN GENERAL: To solve a complicated inequality:

- **STEP 1:** Solve the corresponding **equation**.
- **STEP 2:** Graph both sides of the equation.

  NOTE: Your answers to STEP 1 should correspond to intersection points of the graphs.
- **STEP 3:** Solve the inequality using your graph and the intersection points.

EXAMPLE: Use the process above to solve $x^2 < x+6$.

- **STEP 1:** Solve $x^2 = x+6$.

- **STEP 2:** Graph $f(x) = x^2$ and $g(x) = x+6$. Check the intersection points match with your solutions from STEP 1.

- **STEP 3:** Use your graph and the intersection points to write the solution to $x^2 < x+6$ using interval notation.

EXAMPLE: Solve $x^2 - 1 \leq 4 - x$.

- **STEP 1:** Solve $x^2 - 1 = 4 - x$.

- **STEP 2:** Graph $f(x) = x^2 - 1$ and $g(x) = 4 - x$ to check intersection points.

- **STEP 3:** Solve $x^2 - 1 \leq 4 - x$. Write your answer using interval notation.

CHALLENGE! Solve $x^2 - 2 > |2x + 1|$.

- **STEP 1:** Solve $x^2 - 2 = |2x + 1|$.

  HINT: Start with: $|2x + 1| = x^2 - 2$ means $2x + 1 = x^2 - 2$ or $2x + 1 = -(x^2 - 2)$

- **STEP 2:** Graph $f(x) = x^2 - 2$ and $g(x) = |2x + 1|$ and check intersection points.

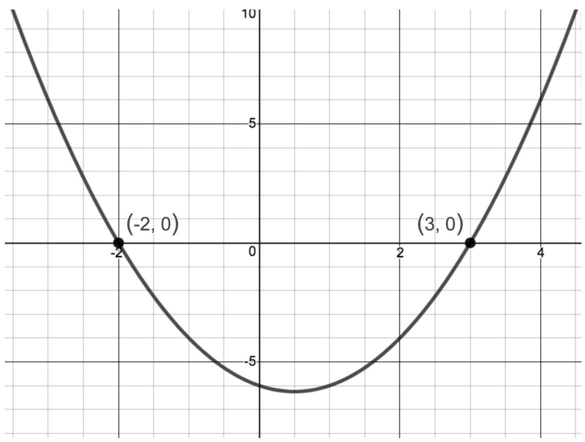
  NOTE: Your graph should show that some of your answers from STEP 1 are extraneous solutions!

- **STEP 3:** Solve $x^2 - 2 > |2x + 1|$. Write your answer using interval notation.

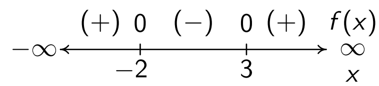
SOLVING INEQUALITIES WITH SIGN DIAGRAMS

In this section we develop a way to solve inequalities algebraically, without the need of graphing technology.

To get started, let's look at the graph of $f(x) = x^2 - x - 6$ below on the left. We have x-intercepts $(-2, 0)$ and $(3, 0)$. Moreover, we see that if $x < -2$, then $f(x) > 0$, if $-2 < x < 3$, $f(x) < 0$, and if $x > 3$, then $f(x) > 0$ once more. We summarize the 'signs' of $f(x)$ ($(+)$, $(-)$, and 0) below on the right in what is called a **Sign Diagram**.



The graph of $f(x) = x^2 - x - 6$

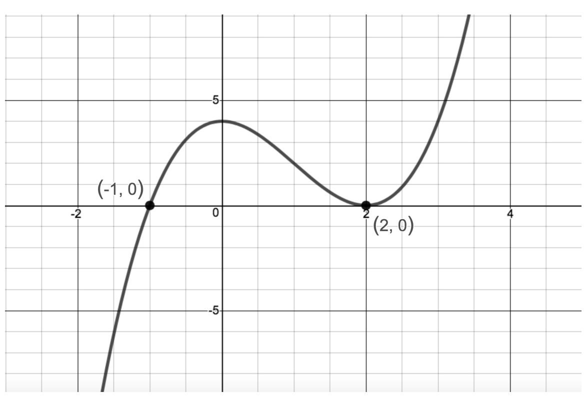


A Sign Diagram for $f(x) = x^2 - x - 6$

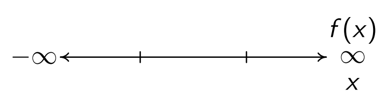
SIGN DIAGRAM: A sign diagram for a function f is a compact way to represent where . . .

- $f(x) = 0$. Represented by '0,' geometrically this is where the graph of f has x-intercepts (i.e., $y = 0$.)
- $f(x) > 0$. Represented by '$(+)$,' geometrically this is where the graph of f is **above** the x-axis (i.e., $y > 0$.)
- $f(x) < 0$. Represented by '$(-)$,' geometrically this is where the graph of f is **below** the x-axis (i.e., $y < 0$.)

EXAMPLE: Use the graph below on the left of $y = f(x)$ to make a sign diagram for $f(x)$.



The graph of $y = f(x)$



A Sign Diagram for $f(x)$

Given a formula for a function $f(x)$ it is possible to construct a sign diagram **without** graphing as follows:

STEPS TO CONSTRUCT A SIGN DIAGRAM WITHOUT GRAPHING:

- **STEP 1:** Solve $f(x) = 0$.

  NOTE: The solutions here are called the **zeros** of f and correspond to the x-intercepts of the graph of $y = f(x)$.

- **STEP 2:** The zeros of f divide the real number line into open intervals.

  Choose one real number (it doesn't matter which one) in each of these intervals – these are called **test values**. [1]

  Evaluate the function at each test value and record the sign of $f(x)$: $(+)$ if $f(x) > 0$; $(-)$ if $f(x) < 0$.

- **STEP 3:** On the real number line, plot the zeros from STEP 1 and place a '0' above them. Between each zero, list the test value you chose under the corresponding interval and place either a '$(+)$' or a '$(-)$' above those intervals.

EXAMPLE: Follow the steps outlined above to make a Sign Diagram for $f(x) = 2x^2+x-6$. Check your answer graphically.

- **STEP 1:** Solve $f(x) = 0$: We solve $2x^2 + x - 6 = 0$ by factoring: $(2x-3)(x+2) = 0$. We get $x = \frac{3}{2}$ or $x = -2$.

- **STEP 2:** $x = \frac{3}{2}$ and $x = -2$ divide the real number line into three intervals: $x < -2$, $-2 < x < \frac{3}{2}$, and $x > \frac{3}{2}$.

  For $x < -2$: we choose $x = -3$ as our test value and find $f(-3) = 2(-3)^2 + (-3) - 6 = 9$, so $f(x)$ is $(+)$ here.

  For $-2 < x < \frac{3}{2}$: we choose $x = 0$ as our test value and find $f(0) = 2(0)^2 + (0) - 6 = -6$, so $f(x)$ is $(-)$ here.

  For $x > \frac{3}{2}$: we choose $x = 2$ as our test value and find $f(2) = 2(2)^2 + (2) - 6 = 4$, so $f(x)$ is $(+)$ here.

- **STEP 3:** Our Sign Diagram is below:

$$
\begin{array}{cccccccc}
 & (+) & 0 & (-) & 0 & (+) & f(x) \\
-\infty \longleftarrow & & | & & | & & \longrightarrow \infty \\
 & \uparrow & -2 & \uparrow & \frac{3}{2} & \uparrow & x \\
 & -3 & & 0 & & 2 &
\end{array}
$$

EXAMPLE: Follow the steps outlined above to make a Sign Diagram for $f(x) = 4 - x^2$. Check your answer graphically.

[1] These numbers are called 'test' values because we're 'testing' the sign of $f(x)$ at these values.

We use Sign Diagrams to solve inequalities as outlined below:

SOLVING INEQUALITIES USING A SIGN DIAGRAM:

- **STEP 1:** (Re)write the inequality to obtain '0' on one side. We'll call the other side '$f(x)$.'
- **STEP 2:** Make a Sign Diagram for f by solving $f(x) = 0$ and using the graph.
- **STEP 3:** Record the solution to the inequality in STEP 1 by examining the Sign Diagram.

EXAMPLE: Solve $2x^2 \geq 6 - x$ using a Sign Diagram. Write your answer using interval notation.

We begin by gathering all the nonzero terms onto one side of the inequality: $2x^2 \geq 6 - x$ becomes $2x^2 + x - 6 \geq 0$.
We identify $f(x) = 2x^2 + x - 6$. We already made a sign diagram for $f(x)$ above, so we record it again below for reference:

$$\begin{array}{ccccccc} & (+) & 0 & (-) & 0 & (+) & f(x) \\ -\infty \longleftarrow & & | & & | & & \longrightarrow \infty \\ & \uparrow & -2 & \uparrow & \frac{3}{2} & \uparrow & x \\ & -3 & & 0 & & 2 & \end{array}$$

To solve $f(x) = 2x^2 + x - 6 \geq 0$ means we are looking for where $f(x) > 0$ (i.e., $(+)$) or where $f(x) = 0$.
We see $f(x)$ is $(+)$ on $(-\infty, -2)$ as well as on $\left(\frac{3}{2}, \infty\right)$. $f(x) = 0$ when $x = -2$ or $x = \frac{3}{2}$.

Putting these together, we get our final answer: $(-\infty, -2] \cup \left[\frac{3}{2}, \infty\right)$.

EXAMPLE: Solve the following inequalities using a Sign Diagram. Check your answers graphically.

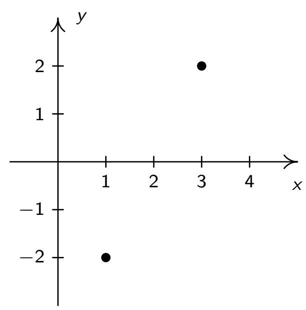
- $x^2 \geq x + 12$

- $x^2 - 4x < 1$

- $x(3 - x) < 3(x + 2)$

FOOTNOTE: Choosing Test Values - a prelude to Calculus:

You may wonder: does it matter which test values I pick? The answer is 'No.' You and a classmate could always choose different test values, but you'd end up with the same Sign Diagram. The mathematics behind this relies on the function under consideration being **continuous**. While this property is studied more deeply in Calculus, for us it suffices to think of a continuous function as one whose graph is 'connected' - that is, there are no holes, jumps, or tears in the graph. In the 'challenge' below, try to draw a continuous function that connects the two points below without crossing the x-axis:



The **Intermediate Value Theorem**, or IVT says that it's impossible to do so. In essence, the IVT says it's impossible for a continuous function to switch signs (from $(-)$ to $(+)$) on an interval without going through 0 somewhere in between. That's why it doesn't matter which test values you pick to construct your Sign Diagram: as long as you've found all the zeros first, the function between the zeros is either always $(+)$ or always $(-)$.

HOMEWORK: Section 1.3: 39 - 55 odd; Section 1.4: 15 - 29 odd.